Unimodular smooth Fano polytopes and their relation with Ewald conditions

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Abstract

Smooth Fano polytopes (SFPs) play an important role in toric geometry and combinatorics. In this paper, we introduce a specific subcollection of them, i.e., the unimodular smooth Fano polytopes (USFPs). In Section 2, they are verified to satisfy the three (weak, strong, star) Ewald conditions. Besides, a characterisation of USFPs is provided as a corollary of the famous Seymour's decomposition theorem. After that, we introduce the previous result by L. Crespo, A. Pelayo and F. Santos, and give a proof of the claim that any deeply monotone polytope is in fact the dual polytope of some USFP. In other words, we extend their results on deeply monotone polytopes to the case of USFPs.

1 Introduction

Smooth Fano polytopes have been intensively studied for several decades. Their unimodular equivalence classes are convex geometric objects corresponding to the ones of smooth toric Fano varieties. Thus, it allows one to pose, consider, and solve problems about them from both combinatorial and algebraic view points. The dual concept of smooth Fano polytopes is known as monotone (moment) polytopes, which correspond to monotone symplectic manifolds. For more details on them on symplectic geometry, we refer to the book by Cannas da Silva [4]. In addition, they also have an emphasis on classical mechanics. See, e.g., [1]. Up to unimodular equivalence, smooth Fano polytopes and monotone polytopes share the same amount for any fixed dimension. Currently, they have been completely classified until dimension 9 [7], [11]. However, it is hard to proceed the process in higher dimension since the number increases rapidly. Meanwhile, people are also interested in searching for possible bounds on invariants. For example, Casagrande showed that the maximal number of vertices of a smooth Fano n-polytope is $3n$ [3]. Besides, there are also bounds on volume and lattce points. An interesting example without smoothness can be found in [10], Theorem B and C. For the smooth case, an attractive open question is the Ewald conjecture [5], which claims a volumn bound for any smooth Fano n-polytope. The most recent partial result of this conjecture is given by Crespo, Pelayo and Santos [2].
For approaching Ewald conjecture, we are motivated to generalize his result to a larger subcollection of smooth Fano polytopes. So, an important goal of this paper is to show that the set of so-called unimodular smooth Fano polytopes is such an extension.

1.1 Smooth Fano Polytopes and Monotone Polytopes

Definition 1. Let $P \subset N_R(\cong \mathbb{R}^n)$ be an n-dimensional lattice polytope, then it is said to be

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(1) *projective* if it contains the origin as its interior;

(2) *Fano* if it's projective and any of its vertices is a primitive lattice point;

(3) *reflexive* if its dual polytope is still a lattice polytope. The dual polytope P^* consists of $x \in \mathbb{R}^n$ s.t. $\langle x, y\rangle \geq -1$ for all $y \in P$, where $\langle x, y\rangle$ is the standard inner product of $\mathbb{R}^n$.

Nowadays, the two 'dual' definitions of smooth lattice polytopes are both commonly used by mathematicians. One starts with the polyhedral cone, whose apex is the origin, over a facet of the polytope, e.g., [11]. The other regards the corner at a fixed vertex of the polytope as a cone, e.g. [2],[8]. Let $N \cong \mathbb{Z}^n$ be a lattice with associated real vector space $N_R \cong N \otimes_{\mathbb{Z}} \mathbb{R}$ and let M be the dual lattice of N with associated real vector spaces $M_R \cong M \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have the two definitions as follows:

Definition 2. Let $P \subset N_R(\cong \mathbb{R}^n)$ be an n-dimensional lattice polytope, then it is said to be

(4a) *simplicial* if there are precisely n vertices for each facet;

(5a) *smooth* if it is simplicial and the vertices of any facet of P form a basis of the lattice N.

Definition 3. Let $Q \subset M_R(\cong \mathbb{R}^n)$ be an n-dimensional lattice polytope, then it is said to be

(4b) *simple* if there are precisely n edges meeting at each fixed vertex;

(5b) *smooth* if it is simple and the primitive edge-direction vectors at each vertex form a basis of the lattice M.

When talking about *smooth Fano polytope*, we always mean the polytope in N_R, i.e., Definition 2. Besides, the smooth reflexive polytope in M_R with Definition 3 is also called a *monotone polytope* (or *reflexive Delzant polytope*), whose name arises from the research of sympletic toric geometry. In most cases where the definition is clear, we simply use $\mathbb{R}^n$ as our vector space for polytopes.

Unimodular equivalence

Let P and P' be two lattice polytopes in $\mathbb{R}^n$, then we say that P and P' are *unimodularly equivalent* if there exists a unimodular transformation which maps P to P'. Here, a *unimodular transformation* is a linear map whose representative matrix is an integer matrix and has the determinant ± 1. All the polytoeps in this paper, if nothing more is mentioned, are up to unimodular equivalence.

Example 1. There are precisely 5 smooth Fano polytopes in dimension 2 up to unimodular equivalence. They are shown in Figure 1 together with their dual polytopes.

1.2 Ewald Conjectures and Ewald Conditions

In the research of smooth Fano polytopes, there is a long-standing open question raised by Gü nter Ewald in 1988 [5]. He conjectured that any n-dimensional smooth Fano polytope can be embedded into $[-1, 1]^n$. After that, Mikkel Øbro [11] mentioned a stronger version of Ewald conjecture and gave a positive answer to the case up to dimension 7. Since it's hard to prove these Ewald conjectures directly, a natural idea is to find a possible collection of polytopes with more restrictions such that they satisfy the condition of these conjectures, and then 'expand' it to the original conjectures. In other words, we are trying to find the maximal subset of smooth

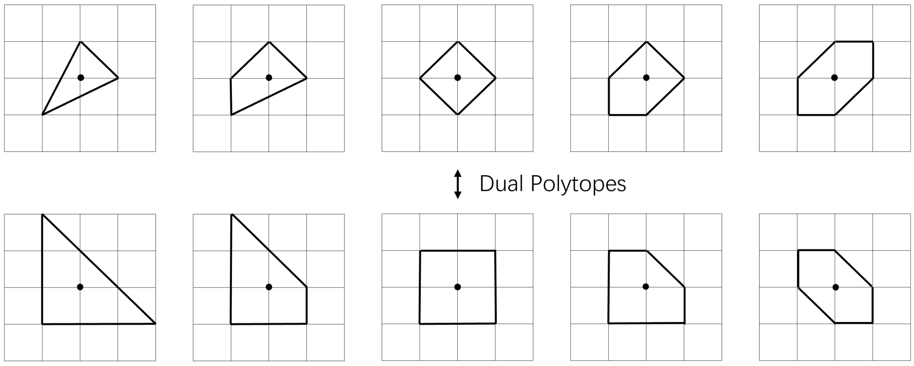

Figure 1: 2-dimensional smooth Fano polytopes and their duals

Fano polytopes that is true for the so-called weak and strong 'Ewald conditions'. Moreover, Dusa McDuff introduced another Ewald condition called star Ewald, inspired by problems in toric symplectic geometry [8]. The concrete expression of these conditions are listed below.

Definition 4. Let P be an n-dimensional projective polytope, then we say that

- it satisfies *the weak Ewald condition* if P can be embedded into the hypercube $[-1,1]^n$ via a unimodular transformation;
- equivalently, it satisfies *the weak Ewald condition* if the symmetric point set $\mathcal{E}(P^*)$ of its dual polytope P^* contains a unimodular basis of $\mathbb{Z}^n$, where the *symmetric point set* is defined to be
$$\mathcal{E}(P^*) := P^* \cap (-P^*) \cap \mathbb{Z}^n;$$
- it satisfies *the strong Ewald condition* if for any vertex $v \in P$, there exists a unimodular transformation ϕ, s.t. $\phi(P) \subset [-1,1]^n$ and $\phi(v) = \sum_{i=1}^{n} e_i$, where e_i are standard basis vectors of $\mathbb{R}^n$;
- equivalently, it satisfies *the strong Ewald condition* if for any facet F of its dual polytope P^*, the set $\mathcal{E}(P^*) \cap F$ contains a unimodular basis of $\mathbb{Z}^n$;
- P^* satisfies *the star Ewald condition* if every face of P^* satisfies the star Ewald condition. A face f of P^* is said to be *star Ewald* if there exists a symmetric point $\lambda \in \mathcal{E}(P^*)$ s.t. $\lambda \in \mathrm{Star}^*(f)$ and $-\lambda \notin \mathrm{Star}(f)$. Here, $\mathrm{Star}(f)$ is the union of all facets (face of codimension 1) containing face f, and $\mathrm{star}(f)$ is the union of all ridges (face of codimension 2) containing face f. Besides, $\mathrm{Star}^*(f) := \mathrm{Star}(f) \backslash \mathrm{star}(f)$.

Furthermore, Benjamin Nill generalized the weak Ewald conjecture in 2009 to the case where reflexivity of the dual polytope is not required [9]:

Conjecture 1 (Generalized Ewald conjecture [9]). *Let $P \subset N_{\mathbb{R}}$ be an n-dimensional polytope whose dual polytope P^* is a smooth projective lattice polytope in $M_{\mathbb{R}}$, then P lies in the hypercube $[-1,1]^n$.*

A recent result in dimension 2 and some partial results in dimension 3 have been verified by Luis Crespo [2]. Though it seems more general without the requirement of reflexivity, Nill's conjecture is believed to be implied by the weak Ewald conjecture [2].

2 Main Results

DEFINITION 5. Let $P \subset \mathbb{R}^n$ be a lattice polytope, then the (coordinate) matrix of P is an $m \times n$-matrix ($m > n$) whose each row is the coordinates of a vertex. The polytope P is said to be *unimodular* if any n vertices form either a basis of the lattice or are linearly dependent. In other words, any $n \times n$-submatrix has determinant ± 1 or 0.

There are three main theorems in this paper:

THEOREM 2.1. *Let P be an n-dimensional unimodular smooth Fano polytope with m vertices, then*

(i) P satisfies the strong Ewald condition, and

(ii) the dual polytope P^* of P satisfies the star Ewald condition.

THEOREM 2.2. *Let P be a unimodular smooth Fano polytope. Then its coordinate matrix M can be constructed from graphic matrices by elementary operations, (matroid-) dualizing and k-sums for $k = 1, 2, 3$.*

REMARK 1. This result can be seen as a corollary of the famous Seymour's decomposition theorem on matroids [13]. In addition, for any given graphic matrices, one may construct a lattice polytope on it. If the polytope is smooth Fano, then we call it a *smooth Fano polytope arising from directed graph* (SFPdG for short, and details can be found in [6]). Obviously, each SFPdG is a USFP.

DEFINITION 6. An n-dimensional monotone polytope P is said to be *deeply (smooth)* if the paralledpiped at any vertex v of P, i.e.,

$$\{v + \sum_{i=1}^{n} \lambda_i u_i |\ \lambda_i \in [0, 1],\ \forall i\}$$

is contained in P.

THEOREM 2.3. *Any deeply monotone polytope is the dual polytope of a unimodular smooth Fano polytope.*

REMARK 2. With the results above, we have the following relations:

$$\begin{array}{ccc} \{\text{Deeply monotone polytopes}\} & \subset & \{\text{Duals of USFPs}\} \\ \cap & & \cup \\ \{\text{UT-free monotone polytopes}\} & & \{\text{Duals of SFPdGs}\} \end{array}$$

Here, a *UT-free monotone polytope* is a monotone polytope without any unimodular triangle as a 2-face [2].

Now, we know that USFPs satisfy three Ewald conditions by Theorem 2.1, and then, DMP and SFPdG follow from the inclusions. Naturally, one may conjecture:

CONJECTURE 2. *UT-free monotone polytopes are the duals of USFPs (up to unimodular equivalence). In particular, all the UT-free monotone polytopes satisfy the three Ewald conditions.*

REMARK 3. Furthermore, we can replace the 'USFPs' in Conjecture 2 by the 'SFPdGs', i.e.,

$$\{\text{UT-free monotone polytopes}\} \subset \{\text{Duals of SFPdGs}\}.$$

which is still very likely to be true.

REMARK 4. The numbers of these types of polytopes, up to dimension 6 are listed below:

Dimension	SFPs	UT-free	DMPs	SFPdGs	USFPs
2	5	5	5	5	5
3	18	16	16	16	16
4	124	74	72	95	96
5	866	336	300	551	554
6	7622	1699	1352	≥ 3920	4097

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